Formally Verified Approximations of Definite Integrals

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Techniques for Reliable Enclosures of Functions Automating the proof process Results in Practice Perspectives

Numerical Integrals in Modern Math Proofs Problem Description

Numerical Integrals in Modern Math Proofs

- **Double bubbles minimize** (Hass, Hutchings, Schlafly 1995): "The proof parameterizes the space of possible solutions by a two-dimensional rectangle [...]. This rectangle is subdivided into 15,016 smaller rectangles which are investigated by calculations involving a total of 51,256 numerical integrals."
- Ternary Goldbach Conjecture (Helfgott 2013):

$$\int_{-\infty}^{\infty} \frac{(0.5 \cdot \log(\tau^2 + 2.25) + 4.1396 + \log \pi)^2}{0.25 + \tau^2}$$

"We compute the last integral numerically (from -100,000 to 100,000)".

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Numerical Integrals in Modern Math Proofs Problem Description



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Rigorous numerical integration

I need to evaluate some (one-variable) integrals that neither SAGE nor Mathematica can do symbolically. As far as I can tell, I have two options:

(a) Use GSL (via SAGE), Maxima or Mathematica to do numerical integration. This is really a non-option, since, if I understand correctly, the "error bound" they give is not really a guarantee.



9

(b) Cobble together my own programs using the trapezoidal rule, Simpson's rule, etc., and get rigorous error bounds using bounds I have for the second (or fourth, or what have you) derivative of the function I am integrating. This is what I have been doing.

Is there a third option? Is there standard software that does (b) for me?

na.numerical-analysis

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Results in Practice

Integrals.

Numerical Integrals in Modern Math Proofs

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Most often, integral estimation \neq symbolic resolution e.g., Rump Integral:

$$\int_0^8 \sin(x + \exp x) dx$$

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Integrals

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Most often, integral estimation \neq symbolic resolution e.g., Rump Integral:

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We need numerical methods to get estimates.

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Numerical Integrals in Modern Math Proofs Problem Description

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Problem Description

Compute

$$A \leq \int_u^v f(t) \, dt \leq B$$

knowing:

- $f : \mathbb{R} \to \mathbb{R}$ Riemann-integrable on [u, v];
- [u, v] compact interval of \mathbb{R} ;
- interval extension and/or polynomial approximation of f...

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$$\int_{0}^{1} \frac{\arctan\sqrt{x^{2}+2}}{\sqrt{x^{2}+2} (x^{2}+1)} dx$$

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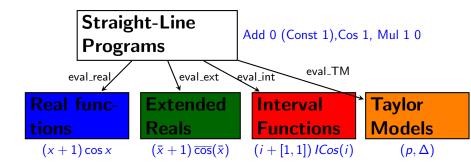
$$\int_{0}^{1} \frac{\arctan \sqrt{x^{2}+2}}{\sqrt{x^{2}+2} (x^{2}+1)} dx$$

... and *automatically* build a proof of this enclosure **Bonus: automatically deduce integrability as well**

Building the extension







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Rigorous is Not Enough

Even Rigorous Methods Can Fail

In 2013, H. Helfgott asks for a rigorous integration tool on Mathoverflow.

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	New analysis was an example of the second se				

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One of the integrals he gives as an example is

$$\int_{0}^{1} \left| \left(x^{4} + 10x^{3} + 19x^{2} - 6x - 6 \right) \exp x \right| dx \simeq 11.14731055005714 \text{ (Coq)}$$

The selected answer is INTLAB (Rump, INTerval LABoratory). INTLAB gives (until May 2016 :-)) 11.147 68687134154 without warning when asked for absolute precision 10^{-15} . Other quadrature methods fail.

Rigorous is Not Enough

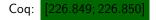
Mistake in the proof of the Ternary Goldbach Conjecture

$$\begin{split} \sqrt{\int_{-\frac{1}{2}-i\infty}^{z} \left|\frac{L(s,\chi)}{L(s,\chi)} \cdot \frac{1}{s}\right| \ |ds| &\leq \sqrt{\int_{-\frac{1}{2}-i\infty}^{z} \left|\frac{\log q}{s}\right| \ |ds|} \\ &+ \sqrt{\int_{-\infty}^{\infty} \frac{\left|\frac{1}{2}\log\left(\tau^{2} + \frac{q}{4}\right) + 4.1396 + \log \pi\right|^{2}}{\frac{1}{4} + \tau^{2}} d\tau} \\ &\leq \sqrt{2\pi}\log q + \sqrt{226.844}, \end{split}$$

where we compute the last integral numerically $\frac{1}{2}$

⁴By a rigorous integration from $\tau = -100000$ to $\tau = 100000$ using VNODE-LP [Ned06], which runs on the PROFIL/BIAS interval arithmetic package [Knü99].





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The End

Thank you for your attention! Any questions?

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Computation Time

Ahmed's integral (Mathematical Spectrum, 2015)

$$\int_{0}^{1} \frac{\arctan\sqrt{x^{2}+2}}{\sqrt{x^{2}+2} (x^{2}+1)} dx = \frac{5\pi^{2}}{96}$$

Error	Time	Accuracy	Degree	Depth	Prec
10^{-3}	0.5	9	5	1	30
10^{-6}	1.2	19	7	3	30
10^{-9}	2.8	29	7	3	40
10^{-12}	5.5	39	10	3	50
10^{-15}	11.2	49	10	4	55

What it looks like in Coq

Prove

$$\left| \int_0^1 \frac{\arctan\sqrt{x^2 + 2}}{\sqrt{x^2 + 2} \ (x^2 + 1)} dx - \frac{5\pi^2}{96} \right| \le 10^{-15}$$

What it looks like in Coq

Prove

$$\left| \int_{0}^{1} \frac{\arctan\sqrt{x^{2}+2}}{\sqrt{x^{2}+2} \ (x^{2}+1)} dx - \frac{5\pi^{2}}{96} \right| \leq 10^{-15}$$

```
Lemma AhmedIntegral :
Rabs (RInt (fun x \Rightarrow
(atan(sqrt(x<sup>2</sup> + 2))) /
((sqrt(x<sup>2</sup> + 2)) * (x<sup>2</sup> + 1))) 0 1 - (5 * PI<sup>2</sup> / 96))
<= 1 / 10^15.
Proof
Time interval with (i_integral_prec 49, i_integral_deg 10,
     i_integral_depth 4, i_prec 55).
(* Finished transaction in 6.357 secs (6.348u,0.004s) (
    successful)
 *)
Qed.
```

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