Formally Verified Approximations of Definite Integrals

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Numerical Integrals in Modern Math Proofs

- **Double bubbles minimize** (Hass, Hutchings, Schlafly 1995): "The proof parameterizes the space of possible solutions by a two-dimensional rectangle [...]. This rectangle is subdivided into 15,016 smaller rectangles which are investigated by calculations involving a total of 51,256 numerical integrals."
- **Ternary Goldbach Conjecture** (Helfgott 2013):

$$\int_{-\infty}^{\infty} \frac{(0.5 \cdot \log(\tau^2 + 2.25) + 4.1396 + \log \pi)^2}{0.25 + \tau^2}$$

"We compute the last integral numerically (from -100,000 to 100,000)".

mathoverflow

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Rigorous numerical integration

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I need to evaluate some (one-variable) integrals that neither SAGE nor Mathematica can do symbolically. As far as I can tell, I have two options:

(a) Use GSL (via SAGE), Maxima or Mathematica to do numerical integration. This is really a non-option, since, if I understand correctly, the "error bound" they give is not really a guarantee.

(b) Cobble together my own programs using the trapezoidal rule, Simpson's rule, etc., and get rigorous error bounds using bounds I have for the second (or fourth, or what have you) derivative of the function I am integrating. This is what I have been doing.

Is there a third option? Is there standard software that does (b) for me?

na.numerical-analysis

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asked Mar 5 '13 at 23:03

H A Helfgott

3,141 • 17 • 61

Integrals

Most often, integral estimation $\neq$ symbolic resolution
e.g., Rump Integral:

$$\int_0^8 sin(x + \exp x) dx$$

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We need numerical methods to get estimates.

Problem Description

Compute

$$A \leq \int_u^v f(t)\, dt \leq B$$

knowing:

- $f : \mathbb{R} \to \mathbb{R}$ Riemann-integrable on $[u, v]$;
- $[u, v]$ compact interval of $\mathbb{R}$;
- interval extension and/or polynomial approximation of f...

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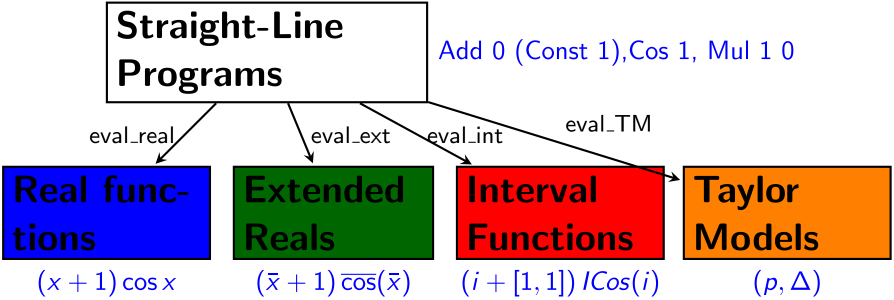
Bonus: automatically deduce integrability as well

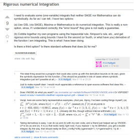
Example for

$$\int_a^b (x+1)\cos x$$



Even Rigorous Methods Can Fail

In 2013, H. Helfgott asks for a rigorous integration tool on Mathoverflow.


Rigorous numerical integration

One of the integrals he gives as an example is

$$\int_0^1 \left|(x^4 + 10x^3 + 19x^2 - 6x - 6)\ \exp x\right| dx \simeq 11.14731055005714 \ \textbf{(Coq)}$$

The selected answer is INTLAB (Rump, INTerval LABoratory). INTLAB gives (until May 2016 :-)) 11.147 68687134154 without warning when asked for absolute precision 10^{-15}. Other quadrature methods fail.

Mistake in the proof of the Ternary Goldbach Conjecture

$$\sqrt{\int_{-\frac{1}{2}-i\infty} \left|\frac{L'(s,\chi)}{L(s,\chi)}\cdot\frac{1}{s}\right|^2 |ds|} \leq \sqrt{\int_{-\frac{1}{2}-i\infty} \left|\frac{\log q}{s}\right|^2 |ds|} + \sqrt{\int_{-\infty}^{\infty} \frac{\left|\frac{1}{2}\log\left(\tau^2+\frac{9}{4}\right)+4.1396+\log\pi\right|^2}{\frac{1}{4}+\tau^2} d\tau} \leq \sqrt{2\pi}\log q + \sqrt{226.844},$$

where we compute the last integral numerically.[4]

Again, we use the fact that, by (2.6), $sG_\delta(s)$ is the Mellin transform of

[4] By a rigorous integration from $\tau = -100000$ to $\tau = 100000$ using VNODE-LP [Ned06], which runs on the PROFIL/BIAS interval arithmetic package [Knü99].

Paper: 226.844

Coq: [226.849; 226.850]

The End

Thank you for your attention! Any questions?

Computation Time

Ahmed's integral (Mathematical Spectrum, 2015)

$$\int_0^1 \frac{\arctan \sqrt{x^2+2}}{\sqrt{x^2+2}\,(x^2+1)} dx = \frac{5\pi^2}{96}$$

Error	Time	Accuracy	Degree	Depth	Prec
10^{-3}	0.5	9	5	1	30
10^{-6}	1.2	19	7	3	30
10^{-9}	2.8	29	7	3	40
10^{-12}	5.5	39	10	3	50
10^{-15}	11.2	49	10	4	55

What it looks like in Coq

Prove

$$\left| \int_0^1 \frac{\arctan \sqrt{x^2+2}}{\sqrt{x^2+2}\,(x^2+1)} dx - \frac{5\pi^2}{96} \right| \leq 10^{-15}$$

What it looks like in Coq

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```
Lemma AhmedIntegral :
Rabs (RInt (fun x =>
(atan(sqrt(x^2 + 2))) /
((sqrt(x^2 + 2)) * (x^2 + 1))) 0 1 - (5 * PI^2 / 96))
<= 1 / 10^15.
Proof.
Time interval with (i_integral_prec 49, i_integral_deg 10,
     i_integral_depth 4, i_prec 55).
(* Finished transaction in 6.357 secs (6.348u,0.004s) (
    successful)
 *)
Qed.
```