Formally Verified Approximations of Definite Integrals

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Numerical Integrals in Modern Math Proofs

- **Double bubbles minimize** (Hass, Hutchings, Schlafly 1995): “The proof parameterizes the space of possible solutions by a two-dimensional rectangle [...]. This rectangle is subdivided into 15,016 smaller rectangles which are investigated by calculations involving a total of 51,256 numerical integrals.”

- **Ternary Goldbach Conjecture** (Helfgott 2013):
  \[
  \int_{-\infty}^{\infty} \frac{(0.5 \cdot \log(\tau^2 + 2.25) + 4.1396 + \log \pi)^2}{0.25 + \tau^2} \]
  “We compute the last integral **numerically** (from -100,000 to 100,000)”.

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Rigorous numerical integration

I need to evaluate some (one-variable) integrals that neither SAGE nor Mathematica can do symbolically. As far as I can tell, I have two options:

(a) Use GSL (via SAGE), Maxima or Mathematica to do numerical integration. This is really a non-option, since, if I understand correctly, the "error bound" they give is not really a guarantee.

(b) Cobble together my own programs using the trapezoidal rule, Simpson's rule, etc., and get rigorous error bounds using bounds I have for the second (or fourth, or what have you) derivative of the function I am integrating. This is what I have been doing.

Is there a third option? Is there standard software that does (b) for me?
Most often, integral estimation \( \neq \) symbolic resolution
e.g., Rump Integral:

\[
\int_0^8 \sin(x + \exp x) \, dx
\]
Most often, integral estimation $\not=$ symbolic resolution

e.g., Rump Integral:

$$\int_0^8 \sin(x + \exp x) \, dx$$

We need numerical methods to get estimates.
Problem Description

Compute

\[ A \leq \int_{u}^{v} f(t) \, dt \leq B \]

knowing:

- \( f : \mathbb{R} \to \mathbb{R} \) Riemann-integrable on \([u, v]\);
- \([u, v]\) compact interval of \(\mathbb{R}\);
- interval extension and/or polynomial approximation of \(f\)...
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\[ \int_{0}^{1} \frac{\arctan \sqrt{x^2 + 2}}{\sqrt{x^2 + 2} (x^2 + 1)} \, dx \]
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\]

... and automatically build a proof of this enclosure
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\[ A \leq \int_{u}^{v} f(t) \, dt \leq B \]

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... and automatically build a proof of this enclosure

**Bonus:** automatically deduce integrability as well
Example for

\[ \int_a^b (x + 1) \cos(x) \]

**Straight-Line Programs**

- **Real functions**
  - \((x + 1) \cos(x)\)
- **Extended Reals**
  - \((\bar{x} + 1) \cos(\bar{x})\)
- **Interval Functions**
  - \((i + [1, 1]) \text{ICos}(i)\)
- **Taylor Models**
  - \((p, \Delta)\)

- **Add 0 (Const 1), Cos 1, Mul 1 0**

- **Eval real**
- **Eval ext**
- **Eval int**
- **Eval TM**
Even Rigorous Methods Can Fail

In 2013, H. Helfgott asks for a rigorous integration tool on Mathoverflow.

One of the integrals he gives as an example is

\[
\int_0^1 |(x^4 + 10x^3 + 19x^2 - 6x - 6) \exp x| \, dx \simeq 11.14731055005714 \quad \text{(Coq)}
\]

The selected answer is INTLAB (Rump, INTerval LABoratory). INTLAB gives (until May 2016 :) ) 11.14768687134154 without warning when asked for absolute precision $10^{-15}$. Other quadrature methods fail.
Mistake in the proof of the Ternary Goldbach Conjecture

\[
\int_{-\frac{1}{2} - i\infty}^{\frac{1}{2} - i\infty} \left| \frac{L(s, \chi)}{L(s, \chi)} \cdot \frac{1}{s} \right| |ds| \leq \int_{-\frac{1}{2} - i\infty}^{\frac{1}{2} - i\infty} \frac{\log q}{s} |ds| + \int_{-\infty}^{\infty} \frac{\frac{1}{2} \log \left( \tau^2 + \frac{9}{4} \right) + 4.1396 + \log \pi^2}{\frac{1}{4} + \tau^2} d\tau \\
\leq \sqrt{2\pi \log q} + \sqrt{226.844},
\]

where we compute the last integral numerically.\(^4\)

Again, we use the fact that, by \(\mathcal{M}\), \(\pi_G(s)\) is the Mellin transform of \(c(s)\).

\(^4\)By a rigorous integration from \(\tau = -100000\) to \(\tau = 100000\) using VNODE-LP [Ned00], which runs on the PROFIL/BIAS interval arithmetic package [Knu94].

Paper: 226.844  Coq: [226.849; 226.850]
Thank you for your attention! Any questions?
Ahmed's integral (Mathematical Spectrum, 2015)

\[ \int_{0}^{1} \frac{\arctan \sqrt{x^2 + 2}}{\sqrt{x^2 + 2} (x^2 + 1)} \, dx = \frac{5\pi^2}{96} \]

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What it looks like in Coq

Prove

\[ \left| \int_{0}^{1} \frac{\arctan \sqrt{x^2 + 2}}{\sqrt{x^2 + 2} (x^2 + 1)} \, dx - \frac{5\pi^2}{96} \right| \leq 10^{-15} \]
What it looks like in Coq

Prove
\[
\left| \int_0^1 \frac{\arctan \sqrt{x^2 + 2}}{\sqrt{x^2 + 2} (x^2 + 1)} \, dx - \frac{5\pi^2}{96} \right| \leq 10^{-15}
\]

Lemma AhmedIntegral :
Rabs (RInt (fun x =>
(atan(sqrt(x^2 + 2))) /
((sqrt(x^2 + 2)) * (x^2 + 1))) 0 1 - (5 * PI^2 / 96))
<= 1 / 10^15.
Proof.
Time interval with (i_integral_prec 49, i_integral_deg 10,
i_integral_depth 4, i_prec 55).
(* Finished transaction in 6.357 secs (6.348u,0.004s) (*
successful)
*)
Qed.